

## APPENDIX B

### GENERALISED SIMPSON'S RULE

This can be used to estimate a multiple integral over any shaped region  $R$  that can be approximated by a collection of  $n$ -dimensional rectangles.

Suppose  $u(x_1, x_2, \dots, x_n)$  is a function of  $n$  variables and we wish to estimate  $\iint_R u(x_1, x_2, \dots, x_n)$ .

Approximate the region by a collection of  $n$ -dimensional rectangles that we call **cells**.

Divide each cell into  $2^n$  **sub-cells** by splitting each cell in half in each of the  $n$ -dimensions. Suppose that the length of each sub-cell is  $p_i$  in the  $i$ 'th direction.

For each cell consider the  $2n + 1$  points where in all but one dimension the point is common to two sub-cells. In the other dimension it may be on the boundary of the whole cell or in the very middle. Let the ordinates at all of these points, for all the cells, be denoted by  $\Gamma$ .

For example if a cell was translated so that it was symmetrically placed around the origin, these points would be:

- $(0, 0, \dots, 0, p_i, 0, \dots, p_n)$  for  $i = 1$  to  $n$  and where the  $p_i$  is the  $i$ 'th coordinate;
- $(0, 0, \dots, 0, -p_i, 0, \dots, p_n)$  for  $i = 1$  to  $n$  and where the  $-p_i$  is the  $i$ 'th coordinate;
- $(0, 0, \dots, 0)$ .

Call the ordinates at the points in the middle of a cell **middle ordinates**. All the other ordinates in  $\Gamma$  can be further classified as internal ordinates or boundary ordinates, according to whether the point lies on the boundary of the total collection of cells or lies inside and therefore lies in two adjoining cells. Note that the middle ordinates are not counted as internal even though they lie inside the collection of cells.

**Theorem (Cooper):**

The estimate of  $\iint_{\mathbb{R}} u(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$  is:

$$\frac{2^{n-1}}{3} p_1 p_2 \dots p_n [2(\Sigma \text{ internal ordinates}) \\ + (\Sigma \text{ boundary ordinates}) \\ - (2n - 6)\Sigma \text{ middle ordinates}].$$

**Case  $n = 1$ :** Here the cells are pairs of strips and the sub-cells are the strips themselves. The boundary ordinates are the end ordinates, which receive a weight of 1. The internal ordinates are the other even ordinates, which receive weight of 2. The middle ordinates are the odd ordinates and, since  $-(2n - 6) = 4$ , they receive a weight of 4 as in the standard one-dimensional Simpson's Rule.

**Case  $n = 2$ :** Here the cells are ordinary rectangles. A cell will be made up of 4 sub-cells. Here  $2n - 6 = -2$  so the middle points have the same weight as the boundary

points, which is why in this case we can include the middle points along with the internal points.

**Case  $n = 3$ :** Here each cell is divided into 8 sub-cells. Since  $2n - 6 = 0$  we can dispense with the middle points.

**Proof of this Theorem:**

We approximate the function by a quadratic function. With  $n$  dimensions this will have the form:

$$v(x_1, x_2, \dots, x_n) = \sum a_i x_i^2 + \sum b_{ij} x_i x_j + \sum c_i x_i + d.$$

$$\int_{-p_i}^{p_i} x_i^2 dx_i = \left[ \frac{1}{3} x_i^3 \right]_{-p_i}^{p_i} = \frac{2}{3} p_i^3.$$

Hence  $\iint_C a_i x_i^2 = \frac{2^n}{3} p_1 p_2 \dots p_n (a_i p_i^2)$  where  $C$  is the cell

$$[-p_1, p_1] \times [-p_2, p_2] \times \dots \times [-p_n, p_n].$$

$$\int_{-p_i}^{p_i} x_i x_j dx_i = \left[ \frac{1}{2} x_i^2 x_j \right]_{-p_i}^{p_i} = 0 \text{ if } i \neq j \text{ so } \iint_C b_{ij} x_i x_j = 0.$$

$$\int_{-p_i}^{p_i} x_i dx_i = \left[ \frac{1}{2} x_{ij} \right]_{-p_i}^{p_i} = 0 \text{ so } \iint_C c_i x_i = 0.$$

And, of course,  $\iint_C d = 2^n p_1 p_2 \dots p_n.$

$$\begin{aligned} \text{Hence } \iint_C [\sum a_i x_i^2 + \sum b_{ij} x_i x_j + \sum c_i x_i + d] \\ = \frac{2^n}{3} p_1 p_2 \dots p_n [\sum a_i p_i^2 + 3d]. \end{aligned}$$

Let  $u(0, 0, \dots, p_i, 0, \dots, 0) = v(0, 0, \dots, p_i, 0, \dots, 0) = u_i^+$   
and

$u(0, 0, \dots, -p_i, 0, \dots, 0) = v(0, 0, \dots, -p_i, 0, \dots, 0) = u_i^-$ ,

where the  $p_i$  and  $-p_i$  are the  $i$ 'th component.

Let  $u(0, 0, \dots, 0) = v(0, 0, \dots, 0) = u_0$ .

Then  $d = u_0$  and  $u_i^+ + u_i^- - 2u_0 = 2a_i p_i^2$ .

Hence  $\iint_C [\Sigma a_i x_i^2 + \Sigma b_{ij} x_i x_j + \Sigma c_i x_i + d]$

$$= \frac{2^{n-1}}{3} p_1 p_2 \dots p_n [\Sigma u_i^+ + \Sigma u_i^- - 2n u_0]$$

$$+ 2^n p_1 p_2 \dots p_n u_0.$$

$$= \frac{2^{n-1}}{3} p_1 p_2 \dots p_n [\Sigma u_i^+ + \Sigma u_i^- - (2n - 6)u_0].$$

Adding these estimates over all the cells:

$\iint_R u(x_1, x_2, \dots, x_n) \approx$

$$\frac{2^{n-1}}{3} p_1 p_2 \dots p_n [2(\Sigma \text{ internal ordinates})$$

$$+ (\Sigma \text{ boundary ordinates})$$

$$- (2n - 6)\Sigma \text{ middle ordinates}].$$

(The extra factor of 2 for the internal points is due to the fact that these belong to two cells and therefore must be counted twice. Note that they are internal to the whole collection of cells but are external to their own cell.)